

TANGENTIAL ALEXANDER POLYNOMIALS AND NON-REDUCED DEGENERATION

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ABSTRACT. We introduce a notion of tangential Alexander polynomials for plane curves and study the relation with θ -Alexander polynomial. As an application, we use these polynomials to study a non-reduced degeneration $C_t \rightarrow D_0 + jL$. We show that there exists a certain surjectivity of the fundamental groups and divisibility among their Alexander polynomials.

1. INTRODUCTION

Let C be a plane curve. We are interested in the geometry of plane curves. Choose a line $L \subset \mathbb{P}^2$ and put $\mathbb{C}_L^2 := \mathbb{P}^2 - L$. As geometrical invariants, we consider

- (a) Fundamental groups: $\pi_1(\mathbb{P}^2 - C)$ and $\pi_1(\mathbb{C}_L^2 - C)$
- (b) Alexander polynomial $\Delta_C(t; L)$.

Zariski studied $\pi_1(\mathbb{P}^2 - C)$ systematically [36] and further developments have been made by many authors. To compute the Alexander polynomial, we need to choose a line at infinity L . However for a generic L , the Alexander polynomial has too much restrictions and we have often the trivial case $\Delta_C(t; L) = (t - 1)^{r-1}$ where r is the number of the irreducible components. In our previous paper [24], we have introduced the notion of θ -Alexander polynomials. This gives more informations for certain reducible curves but it does not give any further information for irreducible curves.

The purpose of this paper is to introduce the notion of the tangential Alexander polynomials. Namely we consider all tangent lines $T_P C$ for the line at infinity. It turns out that tangential Alexander polynomials are related to θ -Alexander polynomials. We apply these polynomials to study non-reduced degenerations. Let $C_t, t \in \Delta$ be a analytic family of reduced curves for $t \neq 0$ such that $C_t \xrightarrow{t \rightarrow \infty} C_0 = D_0 + jL$ where L is a line. The case $j \geq 2$ is a typical *non-reduced degeneration*. In this situation we study the geometry of D_0 using that of C_t . One of our results is the surjectivity assertion of the natural homomorphism:

$$\phi : \pi_1(\mathbb{C}_L^2 - D_0) \rightarrow \pi_1(\mathbb{C}_L^2 - C_t)$$

Here the point is that L is the line component of the limit curve C_0 (Theorem 14, §5). This paper consists of the following sections.

§2 Fundamental groups

§3 Alexander polynomial

§4 Dual stratification and tangential fundamental groups

§5 Degeneration into non-reduced curves with a multiple line

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2. FUNDAMENTAL GROUPS

Let L be a fixed line and put $\mathbb{C}_L^2 := \mathbb{P}^2 - L$. We say L is generic with respect to C if L and C intersects transversely. The topology of $\mathbb{C}_L^2 - C$ does not depend on L if L is generic and we call it the generic affine complement and we often write as \mathbb{C}^2 instead of \mathbb{C}_L^2 . The following Lemma describes the relation of two fundamental groups.

Lemma 1. ([17]) *Let ω be a lasso for L and $N(\omega)$ be the subgroup normally generated by ω .*

(1) *The following sequence is exact.*

$$1 \rightarrow N(\omega) \rightarrow \pi_1(\mathbb{C}_L^2 - C, b_0) \rightarrow \pi_1(\mathbb{P}^2 - C, b_0) \rightarrow 1$$

(2) *Assume that L is generic. Then*

- (i) ω is in the center of $\pi_1(\mathbb{C}_L^2 - C)$ and $N(\omega) \cong \mathbb{Z}$.
- (ii) We have the equality $D(\pi_1(\mathbb{C}^2 - C)) = D(\pi_1(\mathbb{P}^2 - C))$ among their commutator groups.

Thus $\pi_1(\mathbb{P}^2 - C)$ is abelian if and only if $\pi_1(\mathbb{C}^2 - C)$ is abelian.

For non-generic line L , $\pi_1(\mathbb{C}_L^2 - C)$ may be non-abelian even if $\pi_1(\mathbb{P}^2 - C)$ is abelian. For example, let $C = \{Y^2Z - X^3 = 0\}$ and take $L = \{Z = 0\}$. Then $\pi_1(\mathbb{C}_L^2 - C) \cong B_3$ where B_3 is the braid group of three strings and we recall that $B_3 \cong \langle a, b \mid aba = bab \rangle$ ([4]).

2.1. First homology group $H_1(\mathbb{P}^2 - C)$. Assume that C is a projective curve with r irreducible components C_1, \dots, C_r of degree d_1, \dots, d_r respectively. By Lefschetz duality, we have the following.

Proposition 2. *$H_1(\mathbb{P}^2 - C, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{r-1} \times (\mathbb{Z}/d_0\mathbb{Z})$ where $d_0 = \gcd(d_1, \dots, d_r)$. In particular, $H_1(\mathbb{C}_L^2 - C) \cong \mathbb{Z}^r$.*

Take a lasso g_i for each component C_i of C for $i = 1, \dots, r$. Then the corresponding homology classes $\{[g_i], i = 1, \dots, r\}$ give free abelian generators of $H_1(\mathbb{C}_L^2 - C)$.

2.2. Degenerations and fundamental groups. Let C be a reduced plane curve. The total Milnor number $\mu(C)$ is defined by the sum of the local Milnor numbers $\mu(C, P)$ at the singular points P of C . Let $\Delta := \{\zeta \in \mathbb{C} \mid |\zeta| \leq 1\}$ the unit disk. We consider an analytic family of projective curves $C_t = \{F_t(X, Y, Z) = 0\}$, $t \in \Delta$ where $F_t(X, Y, Z)$ are reduced homogeneous polynomial of degree d for any t . We call $\{C_t; t \in \Delta\}$ a reduced degeneration. We assume that C_t , $t \neq 0$ have the same configuration of singularities so that they are topologically equivalent but C_0 may obtain more singularities, i.e., $\mu(C_t) \leq \mu(C_0)$.

Theorem 3. *For a given reduced degeneration $\{C_t; t \in \Delta\}$, there is a canonical surjective homomorphism for $t \neq 0$:*

$$\varphi : \pi_1(\mathbb{P}^2 - C_0) \twoheadrightarrow \pi_1(\mathbb{P}^2 - C_t)$$

In particular, if $\pi_1(\mathbb{P}^2 - C_0)$ is abelian, so is $\pi_1(\mathbb{P}^2 - C_t)$.

See for example, [24] and also Theorem 14 of §5 for another simple proof.

2.3. Product formula. Assume that C_i is a curve of degree d_i , $i = 1, 2$ which are intersecting transversely at $d_1 d_2$ distinct points. We denote the transversality as $C_1 \pitchfork C_2$. Take a line L such that $L \cap C_1 \cap C_2 = \emptyset$. Note that L need not be generic for C_1 or C_2 .

Theorem 4. (Oka-Sakamoto [28]) *Under the above assumption, we have*

$$\pi_1(\mathbb{C}_L^2 - C_1 \cup C_2) \cong \pi_1(\mathbb{C}_L^2 - C_1) \times \pi_1(\mathbb{C}_L^2 - C_2)$$

For further information about fundamental groups, we refer to [4, 12, 18, 20, 32].

2.4. Example.

2.4.1. Abelian cases. A curve C with small singularities has often commutative fundamental group $\pi_1(\mathbb{P}^2 - C)$. Some examples are here:

- C is a smooth irreducible curve.
- Irreducible curves with only A_1 -singularities (i.e., nodes) by [36, 9, 8, 10, 16, 29], or irreducible curve of degree d with a nodes and b cusps (i.e., A_2) with $6b + 2a < d^2$ ([16]).
- $\pi_1(\mathbb{P}^2 - C)$ (respectively $\pi_1(\mathbb{C}_L^2 - C)$) is abelian for any irreducible curve of degree d if it has a flex of order $\geq d - 3$ in \mathbb{C}_L^2 (resp. of order $d - 2$) ([36]).

Let $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ be a polynomial mapping. Recall that α is a *atypical value at infinity* if the topological triviality at infinity fails at $t = \alpha$ for the family of curves $C_t := \overline{f^{-1}(t)}$ (see [34]).

Proposition 5. ([20]) *Let $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ be a polynomial mapping and assume that 0 is not an atypical value at infinity and $C = f^{-1}(0)$ is smooth in \mathbb{C}^2 . Then $\pi_1(\mathbb{C}^2 - C) \cong \mathbb{Z}$.*

2.4.2. Non-abelian case. Assume that p, q are positive integers greater than 1 and consider the curve

$$C_{p,q} : \quad f_p(X, Y, Z)^q + f_q(X, Y, Z)^p = 0$$

where f_p, f_q are polynomials of degree p, q respectively. $C_{p,q}$ is called a *curve of (p, q) -torus type*. Assume that two curves $\{f_p = 0\}$ and $\{f_q = 0\}$ intersect transversely and there is no other singularities of $C_{p,q}$. Then $\pi_1(\mathbb{P}^2 - C_{p,q}) \cong G(p, q, q)$ and $\pi_1(\mathbb{C}^2 - C) \cong G(p, q)$. In particular, if p, q are coprime, $\pi_1(\mathbb{P}^2 - C_{p,q}) \cong \mathbb{Z}_p * \mathbb{Z}_q$. For the definition of $G(p, q)$ and $G(p, q, r)$, we refer to [18].

2.5. Class formula and flex formula. For the study of curves of low degree, it is often important to know the existence of flex points. Let $d = \text{degree}(C)$, \check{d} be the *degree of the dual curve* \check{C} , let $\Sigma(C)$ be the singular points of C and let $\alpha(C)$ be the *number of the flex points*. Then \check{d} and $\alpha(C)$ are given by the formula:

$$\begin{aligned} \check{d} &= d(d - 1) - \sum_{P \in \Sigma(C)} (\mu(C, P) + m(C, P) - 1) \\ \alpha(C) &= 3d(d - 2) - \sum_{P \in \Sigma(C)} \gamma(C, P) \end{aligned}$$

where $m(C, p)$ is the multiplicity of C at P and $\gamma(C, P)$ is the flex defect of the singularity (C, P) [15, 21]. (In [21], we have denoted $\gamma(C, P)$ as $\delta(C, P)$. To distinguish with δ -genus of the singularity, we change the notation.)

3. ALEXANDER POLYNOMIAL

3.1. General definition. Let X be a finite connected CW-complex and let $\phi : \pi_1(X) \rightarrow \mathbb{Z}$ be a surjective homomorphism. We fix a generator t of the infinite cyclic group \mathbb{Z} . Let Λ be the group ring of \mathbb{Z} . Then Λ is isomorphic to the Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$ and Λ is a principal ideal domain. Consider an infinite cyclic covering $p : \tilde{X} \rightarrow X$ such that $p_{\#}(\pi_1(\tilde{X})) = \text{Ker } \phi$. Then $H_1(\tilde{X}, \mathbb{C})$ has a structure of Λ -module where t acts as the canonical covering transformation. Thus by the structure theorem of modules over a principal ideal domain, we have an identification:

$$H_1(\tilde{X}, \mathbb{C}) \cong \Lambda/\lambda_1 \oplus \cdots \oplus \Lambda/\lambda_n$$

as Λ -modules. We normalize the denominators so that λ_i is a polynomial in t with $\lambda_i(0) \neq 0$ for each $i = 1, \dots, n$. The Alexander polynomial associated to ϕ is defined (see [11]) by the product $\Delta_{\phi}(t) := \prod_{i=1}^n \lambda_i(t)$.

3.2. Alexander polynomials of plane curves. In our situation, we consider a plane curve $C = C_1 \cup \cdots \cup C_r$ where C_1, \dots, C_r are irreducible components of degree d_1, \dots, d_r respectively. Take a line L as the line at infinity and let ϕ_{θ} be the composition

$$\phi_{\theta} : \pi_1(\mathbb{C}_L^2 - C) \xrightarrow{\xi} H_1(\mathbb{C}_L^2 - C, \mathbb{Z}) \cong \mathbb{Z}^r \xrightarrow{\theta} \mathbb{Z}$$

where θ is a surjective homomorphism. Recall that θ is determined by giving an integer n_i to each component C_i such that $\gcd(n_1, \dots, n_r) = 1$. We call n_i the weight for C_i . The Alexander polynomial of C with respect to (L, θ) is defined by $\Delta_{\phi_{\theta}}(t)$ and we denote it as $\Delta_C(t; L, \theta)$.

(1) (Generic case) Assume that L to be generic and $\theta = \theta_{sum}$ where θ_{sum} is defined by the canonical summation $\theta_{sum}(a_1, \dots, a_r) = \sum_{i=1}^r a_i$ (weight 1 for each component.) In this case, we simply write as $\Delta_C(t)$ and call it the generic Alexander polynomial of C , as it does not depend on the choice of a generic L .

(2) If θ is the canonical summation θ_{sum} but L is not generic, we denote it as $\Delta_C(t; L)$, omitting θ . In particular, when L is the tangent line of a smooth point $P \in C$, we call $\Delta_C(t; L)$ the tangential Alexander polynomial at P and we also use the notation $\Delta_C(t; P)$.

(3) If L is generic but θ is not θ_{sum} , we called $\Delta_C(t; L, \theta)$ the θ -Alexander polynomial and we denote it as $\Delta_C(t; \theta)$. In [24] we denoted it by $\Delta_{C,\theta}(t)$, but for the consistency of the notation with (2), we use the notation $\Delta_C(t; \theta)$.

Recall that $(t-1)^{r-1} \mid \Delta_C(t)$ ([24]). Thus this is also the case for $\Delta_C(t; L)$ with any line L , as $\Delta_C(t) \mid \Delta_C(t; L)$. We say that $\Delta_C(t; L)$ is trivial if $\Delta_C(t; L) = (t-1)^{r-1}$.

3.3. Fox calculus. Suppose that G is a group and $\phi : G \rightarrow \mathbb{Z}$ is a given surjective homomorphism. Assume that G has a finite presentation as

$$G \cong \langle x_1, \dots, x_n \mid R_1, \dots, R_m \rangle$$

This corresponds to a surjective homomorphism $\psi : F(n) \rightarrow G$ so that $\text{Ker } \psi$ is normally generated by the words R_1, \dots, R_m where $F(n)$ is a free group of rank n , generated by

x_1, \dots, x_n . Consider the group ring $\mathbb{C}(F(n))$ of $F(n)$ with \mathbb{C} -coefficients. The *Fox differentials* $\frac{\partial}{\partial x_j} : \mathbb{C}(F(n)) \rightarrow \mathbb{C}(F(n))$ for $j = 1, \dots, n$, are additive homomorphisms which are characterized by the following properties.

$$(1) \frac{\partial}{\partial x_j} x_i = \delta_{i,j}, \quad (2) \text{(Leibniz rule)} \quad \frac{\partial}{\partial x_j} (uv) = \frac{\partial u}{\partial x_j} + u \frac{\partial v}{\partial x_j}, \quad u, v \in \mathbb{C}(F(n))$$

The composition $\phi \circ \psi : F(n) \rightarrow \mathbb{Z}$ gives a ring homomorphism $\gamma : \mathbb{C}(F(n)) \rightarrow \mathbb{C}[t, t^{-1}]$. The *Alexander matrix* A is an $m \times n$ matrix with coefficients in $\mathbb{C}[t, t^{-1}]$ and its (i, j) -component is given by $\gamma(\frac{\partial R_i}{\partial x_j})$. Then the *Alexander polynomial* $\Delta_\phi(t)$ is defined by the greatest common divisor of $(n-1) \times (n-1)$ -minors of A . In the case of $G = \pi_1(X)$ for some connected topological space X , this definition coincides with the previous one (Fox [6]).

3.3.1. Examples. We give several examples.

1. Consider the trivial case: $G = \mathbb{Z}^r$ and $\phi = \theta_{sum}$, the canonical one. Then

1-1. $G \cong \mathbb{Z} \cong \langle x_1 \rangle$. Then $\Delta(t) = 1$.

1-2. If $G = \mathbb{Z}^r \cong \langle x_1, \dots, x_r \mid R_{i,j} = x_i x_j x_i^{-1} x_j^{-1}, 1 \leq i < j \leq r \rangle$, we have $\Delta(t) = (t-1)^{r-1}$. This follows from the Fox derivation:

$$\frac{\partial}{\partial x_i} (x_i x_j x_i^{-1} x_j^{-1}) = 1 - x_i x_j x_i^{-1}, \quad \frac{\partial}{\partial x_j} (x_i x_j x_i^{-1} x_j^{-1}) = x_i - x_i x_j x_i^{-1} x_j^{-1}.$$

2. Let $C = \{Y^2 Z - X^3 = 0\}$ and $L_{gen} = \{Z = Y\}$, $L = \{Z = 0\}$. Note that $(0, 1, 0)$ is a flex point of C and L is the flex tangent. Then

$$\pi_1(\mathbb{C}_{L_{gen}}^2 - C) \cong \mathbb{Z}, \quad \Delta_C(t; L_{gen}) = 1$$

$$\pi_1(\mathbb{C}_L - C) = \langle x_1, x_2 \mid x_1 x_2 x_1 = x_2 x_1 x_2 \rangle \cong B_3, \quad \Delta_C(t; L) = t^2 - t + 1$$

3. Let us consider the curve $C = \{Y^2 Z^3 - X^5 = 0\} \subset \mathbb{C}^2$ and $L = \{Z = 0\}$, $M = \{Y = 0\}$. Then $\pi_1(\mathbb{P}^2 - C) \cong \mathbb{Z}/5\mathbb{Z}$ and $\pi_1(\mathbb{C}_L^2 - C) \cong G(2, 5)$ and $\pi_1(\mathbb{C}_M^2 - C) \cong G(3, 5)$. In this case, we get

$$\Delta_C(t) = 1, \quad \Delta_C(t; L) = \frac{(t^{10} - 1)(t - 1)}{(t^5 - 1)(t^2 - 1)}, \quad \Delta_C(t; M) = \frac{(t^{15} - 1)(t - 1)}{(t^5 - 1)(t^3 - 1)}$$

3.4. Weakness of the generic Alexander polynomial $\Delta_C(t)$. The following Lemma describes the relation between the Alexander polynomial and local singularities.

Lemma 6. (Libgober [11]) Let P_1, \dots, P_k be the singular points of C (including those at infinity) and let $\Delta_i(t)$ be the characteristic polynomial of the Milnor fibration of the germ (C, P_i) . Then the generic Alexander polynomial satisfies the divisibility: $\Delta_C(t; L) \mid \prod_{i=1}^k \Delta_i(t)$.

Lemma 7. (Libgober [11]) Let d be the degree of C . Then the Alexander polynomial $\Delta_C(t; L_\infty)$ divides the Alexander polynomial at infinity $\Delta_\infty(t)$. If L_∞ is generic, $\Delta_\infty(t)$ is given by $(t^d - 1)^{d-2}(t - 1)$. In particular, the roots of the generic Alexander polynomial are d -th roots of unity.

Corollary 8. ([11], See also [36]) Assume that C is an irreducible curve of degree d and assume that the singularities are either nodes (i.e., A_1) or ordinary cusp singularities (i.e., A_2). If d is not divisible by 6, the generic Alexander polynomial $\Delta_C(t)$ is trivial.

This implies that there does not exist any non-trivial generic Alexander polynomials of degree n with $n \not\equiv 0 \pmod{6}$, for example, this is the case for cubic, quartic and quintic curves, whose singularities are copies of A_1 or A_2 . However even though *there does exist interesting geometry on these curves*. We will show by examples that certain tangential Alexander polynomial gives non-trivial Alexander invariants and we will give an explanation from viewpoint of non-reduced degeneration in §5.

Another weakness of generic Alexander polynomials is for reducible curves. Let C_1 and C_2 be curves which intersect transversely each other. We take a line L so that L does not contain any points of $C_1 \cap C_2$. Note that L need not be generic for $C_1 \cup C_2$. Theorem 4 says that

$$\pi_1(\mathbb{C}_L^2 - C_1 \cup C_2) \cong \pi_1(\mathbb{C}_L^2 - C_1) \times \pi_1(\mathbb{C}_L^2 - C_2)$$

However the Alexander polynomial $\Delta_{C_1 \cup C_2}(t; L)$ loses these informations. In fact, we have

Theorem 9. ([24]) *Assume that C_1 and C_2 intersect transversely and let $C = C_1 \cup C_2$. Let L be a line such that $L \cap C_1 \cap C_2 = \emptyset$. Then $\Delta_C(t; L) = (t-1)^{r-1}$ where r is the number of irreducible components.*

For further information about Alexander polynomials, we refer to [7, 11, 13, 14, 31]

4. DUAL STRATIFICATION AND TANGENTIAL FUNDAMENTAL GROUPS.

4.1. Dual stratification of curves. Let Σ be a finite set of topological equivalent class of curve singularities and let $\mathcal{M}(\Sigma, d)$ be the configuration space of plane curves of degree d with a fixed singularity configuration Σ . Take two curves $C, C' \in \mathcal{M}(\Sigma, d)$ in the same connected component and two smooth points $P \in C$ and $Q \in C'$. We consider their tangent lines $L = T_P C$, $L' = T_Q C'$. Though the topology of (\mathbb{P}^2, C) and (\mathbb{P}^2, C') are topologically equivalent, this may not be the case for $(\mathbb{P}^2, C \cup L)$ and $(\mathbb{P}^2, C' \cup L')$. To analyze this, we introduce the *dual stratification* $\mathcal{S}(C)$ for $C \in \mathcal{S}(\mathcal{M}(\Sigma, d))$ and $\mathcal{S}(\mathcal{M}(\Sigma, d))$ of $\mathcal{M}(\Sigma, d)$ as follows.

Let $\check{\mathbb{P}}^2$ be the dual projective space. Recall that a point $\alpha = (\alpha_1 : \alpha_2 : \alpha_3) \in \check{\mathbb{P}}^2$ (resp. a point $P = (p_1 : p_2 : p_3) \in \mathbb{P}^2$) can be considered as a line $L_\alpha = \{\alpha_1 X + \alpha_2 Y + \alpha_3 Z = 0\}$ in \mathbb{P}^2 (resp. a line $L_P = \{p_1 U + p_2 V + p_3 W = 0\}$ in $\check{\mathbb{P}}^2$). First take $C \in \mathcal{M}(\Sigma, d)$. Let $\Sigma(C) = \{P_1, \dots, P_k\}$ be the singular points of C . Let $\mathcal{P}(d)$ be the set of partition of the integer d . We consider the mappings $\psi : \check{C} \rightarrow \mathcal{P}(d)$ and $\check{\psi} : C \rightarrow \mathcal{P}(\check{d})$ defined as follows. Let $\alpha \in \check{C}$ (resp. $P \in C$) and let $L_\alpha \cap C = \{R_1, \dots, R_\nu\}$ (resp. $L_P \cap \check{C} = \{S_1, \dots, S_\mu\}$). We define $\psi(\alpha) = \{I(C, L_\alpha; R_i), i = 1, \dots, \nu\}$ where $I(C, L_\alpha; R_i)$ is the local intersection multiplicity. Respectively we define $\check{\psi}(P) = \{I(\check{C}, L_P; S_j); j = 1, \dots, \mu\}$. Note that for a generic line $\alpha \in \check{C}$, L_α is a simple tangent line and therefore $\psi(\alpha) = \{2, 1, \dots, 1\}$. For a generic flex point P , the tangent line $L = T_P C$ gives the partition $\psi(L) = \{3, 1, \dots, 1\}$. A line $\alpha \in \check{C}$ is called an *multi-tangent* line if $\psi(\alpha)$ has at least two members ≥ 2 . A simple bi-tangent line is a typical such line which is simply tangent at two smooth points. A smooth point $P \in C$ is called *tangentially generic* if it is smooth and the tangent line $T_P C$ gives the partition $\{2, 1, \dots, 1\}$. Recall that the Gauss map associated with C , denoted as $G_C : C \rightarrow \check{C}$, is defined by $G_C(P) = T_P C$. Let $\Sigma^{ntg}(C) = \{P_{k+1}, \dots, P_{k+t}\}$ be smooth points which are not

tangentially generic and put $\tilde{\Sigma}(C) = \Sigma(C) \cup \Sigma^{ntg}(C) = \{P_1, \dots, P_{k+t}\}$. The dual stratification $\mathcal{S}(C)$ of C is by definition, $\mathcal{S} := \{C - \tilde{\Sigma}(C), \tilde{\Sigma}(C)\}$. Thus if C is irreducible, $\mathcal{S}(C)$ has one open dense stratum made of tangentially generic points and $k+t$ starata made of isolated points.

4.2. Dual stratification of the configuration space $\mathcal{M}(\Sigma, d)$. Now we consider the dual stratification of $\mathcal{M}(\Sigma, d)$. To distinguish a point in $\mathcal{M}(\Sigma, d)$ and the corresponding curve, we denote points in $\mathcal{M}(\Sigma, d)$ by $\alpha \in \mathcal{M}(\Sigma, d)$ and the corresponding curve by C_α . The configuration of the singularities of the dual curve $\Sigma(\check{C}_\alpha)$ is not unique for $\alpha \in \mathcal{M}(\Sigma, d)$ but it has only finite possible types, say $\Sigma_1^*, \dots, \Sigma_\ell^*$ when we fix the configuration space $\mathcal{M}(\Sigma, d)$. We consider the partition of the configuration space by the following sets:

$$\{\alpha \in \mathcal{M}(\Sigma, d); (\Sigma(\check{C}_\alpha), \mathcal{S}(C_\alpha), \mathcal{S}(\check{C}_\alpha), \psi_\alpha, \check{\psi}_\alpha) \text{ are constant}\}$$

The *dual stratification* $\mathcal{S}(\mathcal{M}(\Sigma, d))$ is defined by the strata which are the connected components of these partitions. Thus for a stratum $M \in \mathcal{S}(\mathcal{M}(\Sigma, d))$, each C_α and \check{C}_α , $\alpha \in M$ have constant dual stratifications.

For a stratum $M \in \mathcal{S}(\mathcal{M}(\Sigma, d))$, we can associate a family of plane curves C_α , $\alpha \in M$ such that the dual family of curves \check{C}_α , $\alpha \in M$ is a family in $\mathcal{M}(\Sigma_j^*, \check{d})$ for some j . Observe that any $\alpha \in M$, the dual stratification $\mathcal{S}(C_\alpha)$ and $\mathcal{S}(\check{C}_\alpha)$ are constant for α by definition. Thus for two $\alpha, \beta \in M$, C_α, C_β are homeomorphic as a stratified sets. More precisely we have

Proposition 10. *Take $\alpha_0 \in M$ and take a point $L_{\alpha_0} \in \check{C}_{\alpha_0}$. This induces a continuous family of lines $L_\alpha \in \check{C}_\alpha$ such that $\psi_\alpha(L_\alpha)$ is constant. Then the topology of the affine pair $(\mathbb{C}_{L_\alpha}^2, C)$ does not depend on $\alpha \in M$. In particular the fundamental group $\pi_1(\mathbb{C}_{L_\alpha}^2 - C)$ does not depend on $\alpha \in M$.*

Proof. Recall that the local topology of $C_\alpha \cup L_\alpha$ at an intersection point P is determined by the local Milnor number $\mu(C_\alpha \cup L_\alpha, P)$ and this is determined by $\mu(C_\alpha, P)$ and the local intersection multiplicity $I(C_\alpha, L_\alpha; P)$. The definition of the dual stratification of $\mathcal{S}(\mathcal{M}(\Sigma, d))$ guarantees the μ -constancy of the family of plane curves $C_\alpha \cup L_\alpha$, $\alpha \in M$ of degree $d+1$. \square

4.3. Tangential fundamental group and Tangential Alexander polynomial. For a line $L \in \check{C}$, we call $\pi_1(\mathbb{C}_L^2 - C)$ the *tangential fundamental group* and $\Delta_C(t; L)$ the *tangential Alexander polynomial*. If $L = T_P C$ for some simple point $P \in C$, we also use the notation $\Delta_C(t; P)$ for $\Delta_C(t; T_P C)$. We also define k -fold Alexander polynomial $\Delta_C(t; P_1, \dots, P_k)$ by $\Delta_{C \cup L_1 \cup \dots \cup L_{k-1}}(t; P_k)$ with $L_j = T_{P_j} C$. It is easy to observe that $\pi_1(\mathbb{C}_L^2 - C)$ and $\Delta_C(t; P)$ are constant on the open (dense if C is irreducible) strata of the dual stratification $\mathcal{S}(C)$. However in general it may give a different polynomial for singular lines $L \in \check{C}$ (they are the images of isolated strata of $\mathcal{S}(C)$ by the Gauss map). We will see some examples later. Thus the tangential Alexander polynomials altogether contain more geometrical informations than the generic Alexander polynomials. The main purpose of this paper is to investigate the property of the tangential Alexander polynomials. Note that if C is irreducible, there is only one choice of θ (up to \pm) but there are many choices for L , even for irreducible C .

4.4. Alexander spectrum. We also consider the set of tangential Alexander polynomials

$$\text{t-AS}(C) := \{\Delta_C(t; P); P \in C\}$$

and we call $\text{t-AS}(C)$ *the tangential Alexander spectrum of C* . There exist at most finite polynomials in the spectrum. In fact, it is bounded by the number of strata of $\mathcal{S}(C)$.

We can also define the *k -fold tangential Alexander spectrum of C* by

$$\text{t-AS}^{(k)}(C) := \{\Delta_C(t; P_1, \dots, P_k); P_j \in C\}$$

It often happens that even when the Alexander spectrum $\text{t-AS}(C)$ is trivial, 2-fold Alexander spectrum $\text{t-AS}^{(2)}(C)$ (or higher one) is not trivial.

4.5. Example. We consider $\mathcal{M}(2A_2 + A_1, 4)$ and $\mathcal{M}(E_6, 4)$. By class formula, the dual curve \check{C} of a generic member C of $\mathcal{M}(2A_2 + A_1, 4)$ or $\mathcal{M}(E_6, 4)$ is a quartic with $2A_2 + A_1$ in both cases. (C has generically 2 flex points.) In both configuration spaces $\mathcal{M}(2A_2 + A_1, 4)$ and $\mathcal{M}(E_6, 4)$, there are strata which correspond to *degenerated* members, namely curves with one flex of order 2. This implies that the dual curve has an E_6 singularity. For these configuration spaces, there is a beautiful work by C.T.C.Wall [35]. Consider the subsets:

$$\begin{aligned} M_1 &:= \{C \in \mathcal{M}(2A_2 + A_1, 4); \Sigma(\check{C}) = \{2A_2 + A_1\}\}, \\ M_2 &:= \{C \in \mathcal{M}(2A_2 + A_1, 4); \Sigma(\check{C}) = \{E_6\}\} \\ N_1 &:= \{C \in \mathcal{M}(E_6, 4); \Sigma(\check{C}) = \{2A_2 + A_1\}\}, \\ N_2 &:= \{C \in \mathcal{M}(E_6, 4); \Sigma(\check{C}) = \{E_6\}\} \end{aligned}$$

We can easily see that $\{M_1, M_2\}$, $\{N_1, N_2\}$ are respective dual stratifications of the configuration spaces $\mathcal{M}(2A_2 + A_1, 4)$ and $\mathcal{M}(E_6, 4)$. We observe that under the Gauss map,

$-M_1$ and N_2 are self-dual and

$-M_2$ and N_1 are dual each other.

We observe also that $M_2 \subset \partial M_1$ and $N_2 \subset \partial N_1$.

(1-1) We consider quartic $C_1 \in M_1$ with $\Sigma(C_1) = 2A_2 + A_1$ with two flexes. By the class formula, such a curve has a bi-tangent line. As an example, we take:

$$(1) \quad C_1 : \frac{17}{4}y^4 + 8y^3 + 1/4 + 7/2y^2 - 7/2y^2x^2 + 1/4x^4 - 1/2x^2 = 0$$

C_1 has two cusps at $P_1 = (1, 0)$, $P_2 = (-1, 0)$ and one A_1 at $(0, -1)$. Two flexes are at $Q_1 = (10\sqrt{10}/9, 9)$, $Q_2 = (-10\sqrt{10}/9, 9)$. We have a bi-tangent line $y = 1$ which are tangent at $B_1 = (2\sqrt{2}, 1)$, $B_2 = (-2\sqrt{2}, 1)$. For the dual stratification $\mathcal{S}(C_1)$, we have to take two more points $S_1 = (25/7, 8/7)$, $S_2 = (-25/7, 8/7)$ whose tangent lines pass through P_2 and P_1 respectively. Using Zariski-van Kampen pencil method, we can compute $\pi_1(\mathbb{C}_L^2 - C_1)$ as

$$\pi_1(\mathbb{C}_L^2 - C_1) = \langle \xi_1, \xi_2, \xi_3 \mid \xi_1\xi_2\xi_1 = \xi_2\xi_1\xi_2, \xi_2\xi_3\xi_2 = \xi_3\xi_2\xi_3, \xi_1\xi_3 = \xi_3\xi_1, \xi_1\xi_2\xi_3 = \xi_3\xi_2\xi_1 \rangle$$

where $L = \{y = 1\}$. This gives $\Delta_{C_1}(t; L) = t^2 - t + 1$ by Fox calculus. Other tangent lines give the trivial Alexander polynomial. We leave the proof of this assertion as an exercise. Thus $\text{t-AS}(C_1) = \{1, t^2 - t + 1\}$.

(1-2) We consider the following quartic $C_2 \in M_2$ with $\Sigma(C_2) = 2A_2 + A_1$ with a degenerated flex of order 2 at infinity $L = \{Z = 0\}$:

$$C_2 : \quad 1 - 12y + 36y^2 - 32y^3 - 2x^2 + 12x^2y + x^4 = 0$$

$$\pi_1(\mathbb{C}_L^2 - C_2) = \langle \xi_1, \xi_2, \xi_3 | \xi_1\xi_3\xi_1 = \xi_3\xi_1\xi_3, \xi_3\xi_2\xi_3 = \xi_2\xi_3\xi_2, \xi_2\xi_1 = \xi_1\xi_2 \rangle$$

We can also see that $\Delta_{C_2}(t; L) = t^2 - t + 1$. Note that two A_2 singularities are at $(\pm 1, 0)$ and one A_1 is at $(0, 1/2)$. In the dual stratification $\mathcal{S}(C_2)$, there are two more 'singular' points $R_1 = (9, 8)$ and $R_2 = (-9, 8)$ whose tangent line pass through the cusps. However these tangent lines give trivial tangential Alexander polynomials.

(2-1) Consider a quartic $C_3 \in N_1$, defined by $y^3 + x^4 - x^2y^2 = 0$ which has one E_6 -singularity at O . Two flex points are at $(\pm 6\sqrt{6}/5, 36/5)$. The bi-tangent line is given by $y = 4$. By an easy computation, we observe that $L = \{y = 4\}$ gives $\pi_1(\mathbb{C}_L^2 - C_3) \cong B_3$ and $\Delta_{C_3}(t; L) = t^2 - t + 1$. Other tangent Alexander spectra are trivial. So $t\text{-AS}(C_3) = \{1, t^2 - t + 1\}$. The dual stratification has 5 isolated points.

(2-2) Consider the following quartic $C_4 \in N_2$, $f(x, y) = y^3 + x^4 = 0$, with E_6 and one flex of order 2 at $P = (0, 1, 0)$. Take the flex line $Z = 0$ as L . Then

$$\pi_1(\mathbb{C}_L^2 - C) \cong G(3, 4) = \langle \xi_0, \xi_1, \xi_2 | \xi_0 = \omega\xi_1\omega^{-1}, \xi_1 = \omega\xi_2\omega^{-1} \rangle$$

and $\Delta_{C_4}(t; L) = (t^2 - t + 1)(t^4 - t^2 + 1)$ where $\omega = \xi_2\xi_1\xi_0$.

(3) Let $C : f(x, y) = -\frac{1}{2}y^4 - \frac{1}{2} - 3x^2y^2 + y^2 + \frac{3}{2}x^4 - 4x^3 + 3x^2 = 0$ be a 3 cuspidal quartic. As is well-known [36, 19], the generic affine fundamental group is a finite group of order 12, with presentation

$$\pi_1(\mathbb{C}^2 - C) = \langle \xi, \zeta | \xi \zeta \xi = \zeta \xi \zeta, \xi^2 = \zeta^2 \rangle.$$

Though the fundamental group is not abelian, the generic Alexander polynomial is trivial. For $L = \{y = 0\}$ (this is the tangent cone of a cusp and L corresponds to a flex of \check{C}),

$$\pi_1(\mathbb{C}_L^2 - C) = \langle \xi, \zeta | \xi \zeta \xi = \zeta \xi \zeta \rangle = B_3.$$

By the class formula, the dual curve \check{C} is a cubic curve with a node. $\mathcal{S}(C)$ has 3 singular points from $3A_2$ and two 'singular points' from the bi-tangent line. We can also see that $\pi_1(\mathbb{C}_L^2 - C) \cong B_3$ for an arbitrary tangent line L except the bitangent line L_b . The bi-tangent line is given by $x = 2/3$ in this example. By an easy computation, we see that

$$\pi_1(\mathbb{C}_{L_b}^2 - C) \cong \langle \xi_0, \xi_1, \xi_2, \zeta | \xi_0\xi_1\xi_0 = \xi_1\xi_0\xi_1, \xi_1\xi_2\xi_1 = \xi_2\xi_1\xi_2, \xi_2\zeta\xi_2 = \zeta\xi_2\zeta, \zeta = \xi_1^{-1}\xi_0\xi_1 \rangle$$

and we have

Proposition 11. *For the bitangent line L_b , $\Delta_C(t; L_b) = (t^2 - t + 1)^2$. For any other tangent line L , $\Delta_C(t; L) = t^2 - t + 1$. In particular, this implies that $t\text{-AS}(C) = \{t^2 - t + 1, (t^2 - t + 1)^2\}$.*

4.5.1. *Further example.* In the above examples of quartics, the geometry of $C \cup T_P C$ does not change for flexes of the same order. However this is not the case in general.

Consider a fixed mark point $P \in C$. We call (C, P) a *curve with a marked point* P . Two curves with marked points (C, P) and (C', P') are called a *marked Zariski pair* if $\{C \cup T_P C, C' \cup T_{P'} C'\}$ is a Zariski pair. For further information about Zariski pairs, see [1, 2, 3, 24, 25]. In [26], we have shown that for any quintic B_5 with configuration in the next list, there exist two different flex points P, P' such that (B_5, P) and (B_5, P') are marked Zariski pairs.

$$(\#) \quad \left\{ \begin{array}{l} 4A_2, 4A_2 + A_1, A_5 + 2A_2, A_5 + 2A_2 + A_1, E_6 + 2A_2 \\ E_6 + A_5, 2A_5, A_8 + A_2, A_8 + A_2 + A_1, A_{11} \end{array} \right.$$

In fact their generic Alexander polynomials are given as $t^2 - t + 1$, 1 respectively. This implies that the among flexes of these quintics, there are two classes of different topological nature: one class which does not contribute the tangential Alexander spectrum and the other which contributes by $(t^2 - t + 1)$. We give one example. The following quintic $B_5 : f(x, y) = 0$ has A_{11} singularity at the origin and 9 flex points. Among them, the flex at $P = (0, 1)$ is different from others (a flex of torus type). In fact, $B_5 \cup T_P B_5$ is a sextic of torus type [26]. All other flex points gives trivial tangential Alexander polynomial.

$$\begin{aligned} f(x, y) = & -\frac{33}{64}y^5 + \left(\frac{7}{8}x + \frac{129}{64} \right)y^4 + \left(-\frac{5}{4}x^2 - \frac{15}{8}x - 5/2 \right)y^3 + \\ & \left(\frac{15}{8}x^3 + \frac{13}{4}x^2 + x + 1 \right)y^2 + \left(-\frac{3}{4}x^4 - 2x^3 - 2x^2 \right)y + x^5 + x^4 \end{aligned}$$

FIGURE 1. Quintic with A_{11}

4.6. **θ -Alexander polynomials.** To cover the weakness of Alexander polynomials for irreducible curves, we have proposed θ -Alexander polynomials in [24]. First recall that the radical $\sqrt{q(t)}$ of a polynomial $q(t) = \prod_{i=1}^\nu (t - \xi_i)^{\mu_i}$ is defined by $\sqrt{q(t)} := \prod_{i=1}^\nu (t - \xi_i)$. Here $\mu_i \geq 1, \forall i$. The following theorem shows the importance of θ -Alexander polynomial.

Theorem 12. ([24]) Assume that C, C' be reduced curves and we assume further C' is irreducible. For a given integer n , suppose that the surjective homomorphism $\phi_n : \pi_1(\mathbb{C}_L^2 - C \cup C') \rightarrow \mathbb{Z}$ which has weight 1 on each component of C and weight n on C' . Then

(1) $\Delta_{C \cup C'}(t; L, \phi_n)$ is divisible by $\gcd(\Delta_C(t; L) \times (t-1), (t^n-1))$ for $n \neq 0$.

Suppose further that $C \pitchfork C'$ and $\pi_1(\mathbb{C}_L^2 - C') \cong \mathbb{Z}$. Then

(2) $\Delta_C(t; L) \times (t-1)$ is divisible by $\Delta_{C \cup C'}(t; L, \phi_n)$.

(3) $\sqrt{\Delta_{C \cup C'}(t; L, \phi_n)} = \gcd(\Delta_C(t; L) \times (t-1), (t^n-1))$.

(4) In particular, if $\gcd(\Delta_C(t; L) \times (t-1), (t^n-1)) = \Delta_C(t; L)$, we have

$$\Delta_{C \cup C'}(t; L, \phi_n) = \Delta_C(t; L) \times (t-1).$$

Proof. The proof goes exactly as in [24]. Consider the canonical surjective homomorphism: $h : \pi_1(\mathbb{C}_L^2 - C \cup C') \rightarrow \pi_1(\mathbb{C}_L^2 - C) \times \mathbb{Z}$. Consider the presentation.

$$\pi_1(\mathbb{C}_L^2 - C) = \langle g_1, \dots, g_s \mid R_1, \dots, R_k \rangle, \quad s \leq \text{degree } C$$

The homomorphism ϕ_n factors as $\phi_n = \psi \circ h$ where ψ is the surjective homomorphism $\psi : \pi_1(\mathbb{C}_L^2 - C) \times H_1(\mathbb{C}_L^2 - C') \cong \pi_1(\mathbb{C}_L^2 - C) \times \mathbb{Z} \rightarrow \mathbb{Z}$ where the weight of the second factor is n . Note that $\Delta_{C \cup C'}(t; L, \phi_n) = \Delta_{\phi_n}(t)$ in the notation of §3.1. By the above factorization, we have the divisibility: $\Delta_\psi \mid \Delta_{\phi_n}$. Now the calculation of Δ_ψ is done in the exact same manner as in [24]. We use the presentation:

$$\pi_1(\mathbb{C}_L^2 - C) \times \mathbb{Z} = \langle g_1, \dots, g_s, \xi \mid R_1, \dots, R_k, T_i, 1 \leq i \leq s \rangle$$

where $T_i = g_i \xi g_i^{-1} \xi^{-1}$. The key point of the calculation is the following:

$$\frac{\partial T_i}{\partial g_i} = 1 - g_i \xi g_i^{-1} \mapsto 1 - t^n, \quad \frac{\partial T_i}{\partial \xi} = g_i - g_i \xi g_i^{-1} \xi^{-1} \mapsto t - 1$$

Let M be the Alexander matrix of $\theta_{sum} : \pi_1(\mathbb{C}_L^2 - C) \rightarrow \mathbb{Z}$ and let M' be the Alexander matrix of $\psi : \pi_1(\mathbb{C}_L^2 - C) \times \mathbb{Z} \rightarrow \mathbb{Z}$. Then M' is written as

$$M' = \begin{pmatrix} M & \vec{0} \\ N_1 & N_2 \end{pmatrix}$$

where $\vec{0}$ is a zero vector and N_1 is a $s \times s$ -matrix which is explicitly given as $(1-t^n)E_s$ where E_s is the identity matrix of rank s . The vector N_2 takes the form ${}^t(t-1, \dots, t-1)$ where $t-1$ is repeated s times. For any $(s-1) \times (s-1)$ -minor B of M , let \tilde{B} be the $s \times s$ -minor adding $(k+1)$ -th row and the last column. Then $\det \tilde{B} = \det B \times (t-1)$. Thus we have $\Delta_C(t; L) \times (t-1)$ as the common divisor of such $\det \tilde{B}$'s. Also we get $(t^n-1)^s$ by taking a minor from N_1 . We observe also that any determinant of a $s \times s$ -minor which contains at least two rows of (N_1, N_2) is divisible by (t^n-1) . Thus we observe two divisibilities:

$$\Delta_\psi(t) \mid \gcd(\Delta_C(t; L) \times (t-1), (t^n-1)^s), \quad \gcd(\Delta_C(t; L) \times (t-1), (t^n-1)) \mid \Delta_\psi(t)$$

Note that $\Delta_\psi(t) \mid \Delta_{C \cup C'}(t; \phi_n, L)$, by the usual degeneration argument [24]. Thus the first assertion is immediate from the last divisibility. Suppose further that $C \pitchfork C'$ and $\pi_1(\mathbb{C}_L^2 - C') \cong \mathbb{Z}$. Then h is an isomorphism and therefore $\Delta_{C \cup C'}(t; \phi_n, L) = \Delta_\psi(t)$. Thus the assertions (2), (3) follow immediately. The assertion (4) is a result of (1) and (2). \square

4.7. Relations between the tangential Alexander polynomials and θ -Alexander polynomials. Let C be a plane curve of degree d and let $P \in C$ and let $L = T_P C$. We consider the tangential Alexander polynomial $\Delta_C(t; L)$. Let

$$\pi_1(\mathbb{C}_L^2 - C) \cong \langle g_1, \dots, g_d \mid R_1, \dots, R_\ell \rangle$$

be a presentation of $\pi_1(\mathbb{C}_L^2 - C)$ by generators and relations. Take a generic line L_∞ for $C \cup L$ and put $\mathbb{C}^2 = \mathbb{P}^2 - L_\infty$ as usual. Then by Theorem 4, we have

$$\pi_1(\mathbb{C}^2 - C \cup L) = \pi_1(\mathbb{C}_L^2 - C \cup L_\infty) \cong \pi_1(\mathbb{C}_L^2 - C) \times \pi_1(\mathbb{C}_L^2 - L_\infty) \cong \pi_1(\mathbb{C}_L^2 - C) \times \mathbb{Z}$$

and it has a presentation:

$$\begin{aligned} \pi_1(\mathbb{C}^2 - C \cup L) &= \langle g_1, \dots, g_d, h, h_\infty \mid R_1, \dots, R_\ell, T_1, \dots, T_d, S \rangle \\ &= \langle g_1, \dots, g_d, h_\infty \mid R_1, \dots, R_\ell, T_1, \dots, T_d \rangle \end{aligned}$$

where $T_j = h_\infty g_j h_\infty^{-1} g_j^{-1}$ and $S = h h_\infty g_d \dots g_1$. Now the tangential Alexander polynomial $\Delta_C(t; L)$ is associated to the surjective homomorphism

$$\theta_{sum} : \pi_1(\mathbb{C}_L^2 - C) \rightarrow \mathbb{Z} = \langle t \rangle, g_i \mapsto t$$

Let θ_n be the surjective homomorphism with weight n on L

$$\theta_n : \pi_1(\mathbb{C}^2 - C \cup L) \rightarrow \mathbb{Z} = \langle t \rangle, g_i \mapsto t, h \mapsto t^n$$

Now taking $g_1, \dots, g_k, h_\infty$ as generators of $\pi_1(\mathbb{C}^2 - C \cup L) = \pi_1(\mathbb{C}_L^2 - C \cup L_\infty)$, θ_n corresponds to the homomorphism:

$$\eta_{n+d} : \pi_1(\mathbb{C}_L^2 - C \cup L_\infty) \rightarrow \mathbb{Z}, \quad g_i \mapsto t, \quad h_\infty \mapsto t^{-n-d}$$

The last property is the result of the observation: $1 = \theta_n(S) = t^{d+n} \theta_n(h_\infty)$.

Notation. Hereafter we mainly consider the weight like θ_n which has weight one except a line component L in consideration. So we introduce the following notation which is easier to be understood:

$$\Delta_{C \cup L^n}(t) := \Delta_{C \cup L}(t; \theta_n)$$

The upper index n implies that L has weight n . Using this notation, we also write

$$\Delta_{C \cup L_\infty}(t; \eta_{n+d}, L) = \Delta_{C \cup L_\infty^{-n-d}}(t; L).$$

Thus combining the above argument with Theorem 12 we have shown the following.

Theorem 13. *For any integer n , $\Delta_{C \cup L^n}(t) = \Delta_{C \cup L_\infty^{-n-d}}(t; L)$ and we have the divisibility:*

$$\Delta_{C \cup L_\infty^{-n-d}}(t; L) | \Delta_C(t; L) \times (t - 1) \quad \text{and} \quad \gcd(\Delta_C(t; L) \times (t - 1), (t^{n+d} - 1)) | \Delta_{C \cup L_\infty^{-n-d}}(t; L)$$

Furthermore if $\gcd(\Delta_C(t; L) \times (t - 1), (t^{n+d} - 1)) = \Delta_C(t; L)$, we have the equality:

$$\Delta_C(t; L) \times (t - 1) = \Delta_{C \cup L_\infty^{-n-d}}(t; L).$$

4.7.1. Examples. Let C be a quartic with either $2A_2 + A_1$ or E_6 and one flex of order 2. Let L be the flex tangent line. Then we have shown that $\Delta_C(t; L) = t^2 - t + 1$ and $(t^2 - t + 1)(t^4 - t^2 + 1)$ respectively. We can compute generic Alexander polynomials $\Delta_{C \cup L^2}(t)$ of $C \cup L$ as follows. Take a generic line L_∞ .

(1) C is a quartic with $2A_2 + A_1$ and L is the flex tangent line. As $\Delta_C(t; L) = t^2 - t + 1$, we take weight $n = 2$ on L and by Theorem 12, we get

$$\Delta_{C \cup L_\infty^{-6}}(t; L) = \Delta_{C \cup L^2}(t) = (t^2 - t + 1)(t - 1)$$

Let C_t be a family of quartics with $2A_2 + A_1$ with two flex points for $t \neq 0$ and $C_0 = C$. Let L_1, L_2 be two tangent lines at the flex points of C_t . Then $C_t + L_1 + L_2 \rightarrow C + 2L$. Thus the weight 2 on L is canonical. See §5.7.

(2) C is a quartic with E_6 and L is the flex tangent line at a flex of order 2. As $\Delta_C(t; L) = (t^2 - t + 1)(t^4 - t^2 + 1)$, $\gcd(\sqrt{\Delta_C(t; L)}, (t^{12} - 1)) = \Delta_C(t; L)$. Thus we take $n = 8$.

$$\Delta_{C \cup L^8}(t) = \Delta_{C \cup L_\infty^{-12}}(t; L) = (t - 1)(t^2 - t + 1)(t^4 - t^2 + 1)$$

This can be interpreted as $Y^3Z + X^4 = 0$ is a line degeneration of (3,4)-torus curve of degree 12 as $(Y^3Z + X^4)Z^8 = (YZ^3)^3 + (XZ^2)^3$. See §5.4. Note also that

$$\Delta_{C \cup L^2}(t) = \Delta_{C \cup L_\infty^{-6}}(t; L) = (t - 1)(t^2 - t + 1).$$

We can also interpret this equality as a result of a line degeneration of (2,3)-sextics of torus type as $(Y^3Z + X^4)Z^2 = (YZ)^3 + (X^2Z)^2$.

5. DEGENERATION INTO NON-REDUCED CURVES WITH A MULTIPLE LINE

In this section, we study an analytic family of curves C_t , $t \in \Delta$ such that C_0 is not reduced but it has a line component with multiplicity.

5.1. Admissible polydisk. Consider a reduced curve $C \subset \mathbb{P}^2$ which is defined by a polynomial $f(x, y) = 0$ in the affine space $\mathbb{C}_L^2 := \mathbb{P}^2 - L$ where $L = \{Z = 0\}$. (We do not assume the genericity of the line L .) We assume that $f(x, y)$ is a polynomial in y of degree n . The base point of the pencil $\{L_\eta, \eta \in \mathbb{C}\}$ where $L_\eta := \{X - \eta Z = 0\}$ is given by $B = (0, 1, 0)$ in the homogeneous coordinates. Note that $n < d$ if and only if $B \in C$. We say the pencil $\{x = \eta, \eta \in \mathbb{C}\}$ is *base point admissible* (respectively *base point non-admissible*) if $n = d$ (resp. $n < d$). Recall that L_η is a singular line for C if $L_\eta \cap C \cap \mathbb{C}_L^2$ contains some non-transverse intersection point. For the case $n < d$, we also call L_η singular if the number of the points $L_\eta \cap C \cap \mathbb{C}_L^2$ counted with multiplicity is strictly less than n . In this case, we say L_η a *singular line with disappeared points at infinity*. Using Zariski-van Kampen pencil method with respect to the pencil lines $x = \eta$, $\eta \in \mathbb{C}$, we get a presentation

$$\pi_1(\mathbb{C}_L^2 - C) = \langle g_1, \dots, g_n \mid R_1, \dots, R_m \rangle$$

We consider the polydisk $\Delta_{\alpha, \beta} := \Delta_\alpha \times \Delta_\beta$ where $\Delta_\alpha := \{x \mid |x| \leq \alpha\}$ and $\Delta_\beta := \{y \mid |y| \leq \beta\}$ and we consider the following conditions.

- (1) For any $\eta \in \Delta_\alpha$, $L_\eta \cap C \subset \{\eta\} \times \Delta_{\beta/2}$ and L_η does not have any disappeared points at infinity and
- (2) For any $\eta \in \partial\Delta_\alpha$, L_η is not a singular line.
- (3) For any singular line L_η , $\eta \in \Delta_\alpha$.

We say that the polydisk $\Delta_{\alpha,\beta}$ is *admissible for C with respect to L* if it satisfies (1) and (2). Furthermore, we say that the polydisk $\Delta_{\alpha,\beta}$ is *topologically presenting for C with respect to L* if it satisfies (1), (2) and (3). Observe that if $\Delta_{\alpha,\beta}$ is topologically presenting for C with respect to L , the inclusion $\Delta_{\alpha,\beta} - C \cap \Delta_{\alpha,\beta} \hookrightarrow \mathbb{C}_L^2 - C$ is a homotopy equivalence.

5.2. Two non-reduced degenerations. In this section, we focus the following two types of non-reduced degenerations.

Type 1: Line Degenerations. $\{C_t, t \in \Delta\}$ is an analytic family of irreducible curves of degree d and it degenerate into $C_0 = D_0 + jL_\infty$, $j \geq 0$ where D_0 is an irreducible curve of degree $d-j$ and L is a line. We assume also

- (#) there is a point $Q \in L - L \cap D_0$ such that $Q \in C_t$, $\forall t \neq 0$ and the multiplicity of (C_t, Q) is j .

We call such a degeneration *a line degeneration of order j*. L and Q are called the *limit line* and the *base point* of the degeneration respectively. Here j is a non-negative integer. (We are mainly interested in the case $j \geq 2$.) The condition (#) can be weakened as

- (#') there is an analytic family of points $Q_t \in L \cap C_t$ such that $Q_0 \in L - L \cap D_0$ and the multiplicity of (C_t, Q_t) is j .

In fact, under (#'), we may assume that $L = \{Z = 0\}$ and $Q_t = (\alpha(t), 1, 0)$. Then taking a linear change of coordinates $(x, y) \mapsto (x, y + \alpha(t)x)$, we can assume that $Q_t \equiv (0, 1, 0)$. We recall that O. Zariski observed that 3 cuspidal quartic is a non-reduced line degeneration of order 2 from a family of sextics of torus type [36].

Type 2: Flex Degenerations. First we have a family of reduced curves $\{C_t, t \in \Delta\}$ of the same degree. C_0 can be a degeneration if $\mu(C_0) > \mu(C_t)$, $t \neq 0$ or in the same configuration space if $\mu(C_t) = \mu(C_0)$ but in this case, C_0 is in a different stratum of the dual stratification. On C_t , we are given flex lines $L_1(t), \dots, L_k(t)$, $k \geq 2$ such that the family $L_i(t)$ is an analytic family with $L_i(0) = L$ for $i = 1, \dots, k$. We associate a non-reduced degeneration $C_t + L_1(t) + \dots + L_k(t) \rightarrow C_0 + kL$ and we call this family *a flex degeneration*.

5.3. Surjectivity Theorem for line degenerations. Assume first that we have an analytic family of curves $\{C_t, |t| \leq 1\}$ such that C_t is an irreducible curve of degree d and it degenerates into $C_0 = D_0 + jL$ where D_0 is an irreducible curve of degree $d-j$ and L is a line. We assume that $\{C_t, t \neq 0\}$ has $Q = (0, 1, 0)$ as the base point and the multiplicity of C_t at Q is constantly equal to j . Let $F(X, Y, Z, t) = 0$ be the defining homogeneous polynomial of degree d . We assume that $L = \{Z = 0\}$. By the assumption, we have $F(X, Y, Z, 0) = Z^j G(X, Y, Z)$ where degree $G(X, Y, Z) = d-j$. Put $f(x, y, t) := F(x, y, 1, t)$. This is the affine equation of C_t . Note that

- (1) $\deg_{\{x,y\}} f(x, y, t) = d$ for $t \neq 0$
- (2) $\deg_{\{x,y\}} f(x, y, 0) = d-j$ and $\deg_y f(x, y, t) = d-j$ for any t .

The second assertion follows from Bézout theorem and the assumption that C_t has multiplicity j at Q .

Theorem (Surjectivity) 14. *Under the above assumption, there is a canonical surjection*

$$\phi : \pi_1(\mathbb{C}_L^2 - D_0) \rightarrow \pi_1(\mathbb{C}_L^2 - C_\tau), \quad \tau \neq 0, \text{ sufficiently small}$$

Proof. Note that the linear system $L_\eta = \{x = \eta z\}$, $\eta \in \mathbb{C}$ has the base point $Q = (0, 1, 0)$. Suppose that we have chosen a topologically presenting polydisk $\Delta_{\alpha,\beta}$ for D_0 . Let $f(x, y, \tau)$ be the defining affine polynomial for C_τ . As the effect of the non-reducedness disappears when we put $z = 1$ in $F(X, Y, Z, t)$, $f(x, y, t)$ is a analytic family. Write $f(x, y, t)$ as

$$f(x, y, t) = a_{d-j}(x, t)y^{d-j} + \cdots + a_0(x, t)$$

By continuity, we can assume that this polydisk is also admissible for C_τ for some δ and $|\tau| \leq \delta$. Let η_1, \dots, η_ν be the parameters corresponding to the singular lines for D_0 . We take a small positive number ε so that the disks $\Delta_\varepsilon(\eta_i) := \{\eta \mid |\eta - \eta_i| \leq \varepsilon\}$, $i = 1, \dots, \nu$ are disjoint each other and $\Delta_\varepsilon(\eta_i) \subset \Delta_\alpha$. By the assumption, we have $|\eta_j| < \alpha$ for each j . We choose a generic pencil line L_{η_0} and generators g_1, \dots, g_d in this pencil so that we have a presentation:

$$(2) \quad \pi_1(\mathbb{C}_L^2 - D_0) \cong \langle g_1, \dots, g_d \mid R_1, \dots, R_m \rangle$$

For sufficiently small τ , the original singular pencil L_{η_i} for D_0 may splits into several singular lines for the curves C_τ , $\tau \neq 0$ but they are inside $\Delta_\varepsilon(\eta_i)$. Put the corresponding parameters $\eta_{i,1}, \dots, \eta_{i,\nu_i}$. Note that the monodromy relations around L_{η_i} for D_0 is nothing but the product of the monodromy relations around $L_{\eta_{i,s}}$ for $s = 1, \dots, \nu_i$ under a suitable ordering. This is immediate from the topological stability of the pencil restricted on the circle $\partial\Delta_\varepsilon(\eta_i)$. Note also that C_τ may have a singular line L_η such that $|\eta| \rightarrow \infty$ when $\tau \rightarrow 0$. Anyway we can get a presentation by adding several more relations R_{m+1}, \dots, R_{m+n} to (2) (using the same generators g_1, \dots, g_d):

$$\pi_1(\mathbb{C}_L^2 - C_\tau) = \langle g_1, \dots, g_d \mid R_1, \dots, R_m, R_{m+1}, \dots, R_{m+n} \rangle$$

This and (2) implies that there is a canonical surjection $\phi : \pi_1(\mathbb{C}_L^2 - D_0) \rightarrow \pi_1(\mathbb{C}_L^2 - C_\tau)$. \square

Remark 15. *Though we are mainly concerned in the case $j \geq 2$, the assertion for $j = 0$ gives another proof of Theorem 3.*

Corollary 16. *Under the same assumption, we have the divisibility $\Delta_{C_\tau}(t; L) \mid \Delta_{D_0}(t; L)$.*

Taking a generic line L_∞ for C_τ , τ small and $D_0 \cup L$, we apply Theorem 13 to obtain:

Corollary 17. *Under the same assumption,*

$$\sqrt{\Delta_{C_\tau}(t)} \mid \Delta_{D_0 \cup L_\infty^d}(t)$$

Proof. First note that $\Delta_{C_\tau}(t) \mid \Delta_{C_\tau}(t; L)$ and $\Delta_{C_\tau}(t; L) \mid \Delta_{D_0}(t; L)$ by Corollary 15. Secondly $\Delta_{D_0 \cup L_\infty^d}(t; L) = \Delta_{D_0 \cup L^j}(t)$ by Theorem 13. Lastly we have

$$\sqrt{\Delta_{D_0 \cup L_\infty^d}(t; L)} = \gcd(\Delta_{D_0}(t; L) \times (t-1), t^d - 1)$$

The conclusion is now immediate from these observation, as the factor $(t - 1)$ does not appear in $\Delta_{C_\tau}(t)$ by the irreducibility [24]. \square

5.3.1. Examples of line degenerations of order 1. In [23], we have classified configurations of reduced sextics of torus type. Among them, there are sextics of torus type with components $B_5 + L$ where B_5 is an irreducible quintic. In fact, each of them is a degeneration of irreducible sextics of torus type. We give one such example. $D_0 = B_5$ has one A_{11} singularity at $O = (0, 0)$ and $L = \{Z = 0\}$ and it is a flex tangent. A generic irreducible sextic C_t has $[A_{11} + A_5]$ as singularities. By Theorem 14, we know that $\Delta_{C_t}(t; L) \mid \Delta_{D_0}(t; L)$. As $\Delta_{C_t}(t; L)$ is divisible by the generic Alexander polynomial, which is $t^2 - t + 1$ by [22], we conclude $t^2 - t + 1 \mid \Delta_{D_0}(t; L)$.

$$\begin{aligned} D_0 : & \quad x^4y - 8x^2y^2 + 7y^3 - 4x^3y - 2x^2y + y^2 + x^4 + 2x^5 \\ & \quad + 9y^4 - 6x^3y^2 + 2y^2x + 6y^3x = 0 \\ C_s : & \quad (-x^2 + y)^3 + (-x^2 - x^3 + y + 3y^2 + xy + s x^2y)^2 = 0 \end{aligned}$$

5.3.2. Line degenerations of order 2. We consider quartics which are non-reduced line degenerations of sextics. As a quartic D , we can take quartics with configuration (a) $\Sigma(D) = 2A_2$, or (b) $\Sigma(D) = 2A_2 + A_1, 3A_2$ (with an outer singularity A_1 or A_2) or (c) $\Sigma(D) = A_5, E_6$ or $\Sigma(D) = A_2 + A_3, A_6$ (with a wild inner singularity).

We only explain here the case (b) with $\Sigma(D) = 3A_2$. The other case will be explained systematically in §5.4. Three cuspidal quartic is very special. As the degenerated line, we can take any one of a simple tangent line or a tangent cone at a cusp or a unique bi-tangent line. This is not the case for other quartics listed above. The following family gives line degeneration of order 2 of sextics of torus type defined by $f_2^3 + g_3^2 = 0$ into a quartic with 3 A_2 and a line $\{Z = 0\}$ with multiplicity 2.

(a-1) L is a **simple tangent line**. $\{C_s, s \in \Delta\}$ is a family with an outer A_2 singularity at $Q = (0, 1, 0)$ and $L = \{Z = 0\}$ is a simple tangent line of the quartic D_0 . This degeneration is not a line-degeneration of torus curve which we study in the next section.

$$\begin{aligned} D_0 : & \quad y^4 - 2y^2 + 3xy^2 - 3/4x^2y^2 + 1 - 3x + 3x^2 - x^3 = 0 \\ C_s^{(1)} : & \quad (y - 3/2xy - y^3 + sx^3)^2 + (-y^2 + 1 - x)^3 = 0 \end{aligned}$$

(a-2) L is a **tangent cone of an A_2** . $\{C_s^{(2)}, s \in \Delta\}$ is a family with an A_1 singularity at $Q = (0, 1, 0)$ and $L = \{Z = 0\}$ is a tangent cone of an A_2 singularity.

$$\begin{aligned} D_0 : & \quad -3xy^3 + y^4 + 3xy + 1 + 3x^2y^2 - 5/4x^3y + 3/16x^4 - 3/4x^2 - 2y^2 = 0 \\ f_2(x, y, s) : & \quad 1 + (-1/4 + s)x^2 + xy - y^2 \\ g_3(x, y, s) : & \quad -y^3 + 3/2xy^2 + (-3/4x^2 + 1)y + 1/8x^3 \end{aligned}$$

(a-3) L is the **bi-tangent line**. As is observed in Proposition 11, the Alexander polynomial has multiplicity 2 for $(t^2 - t + 1)$. Thus this case is exceptional for the tangential Alexander polynomial of quartic with $3A_2$. To explain this we consider a family of sextics of torus type $\{C_s^{(3)}, s \in \Delta\}$ with $8A_2$, where two A_2 are outer singularities and they are located at at

$R = (0, 5/2, 1)$ and $R' = (-2, 1/2, 1)$. An inner A_2 is at $Q = (0, 7/2, 1)$. The line $L = \{X = 0\}$ is a bi-tangent line and the limit line degeneration. In an affine equation, we can define them as

$$\begin{aligned} D_0 : xy^3 + \frac{1}{256} (19x^2 - 12y^2 + 75x + 75 + 6xy)^2, C_s^{(3)} = \{f_2^3 + g_3^2 = 0\} \\ f_2(x, y, s) = \frac{1}{4(s-1)} (20ysx + 8s^2xy - 8s^3xy - 70s + 4s^3x^2 - 20s^3y - 21s^2 + 21s^3 - 4xy \\ - 4s^2x^2 + 20s^2y - 8y^2s - 4s^2y^2 + 4s^3y^2 + 48ys - 8sx^2 - 48sx + 18s^3x - 18s^2x) \\ g_3(x, y, s) = \frac{1}{16(s-1)^2} (552ysx + 36xy^2s^5 - 90ys^2x^2 + 138ysx^2 - 42ys^3x^2 + 36y^2s^3x - 84y^2xs^4 \\ + 36y^2s^2x - 108y^2sx - 336s^2xy - 96s^3xy - 75x + 516xs^4y + 120x^2s^4y - 228s^5xy - 48s^5x^2y \\ - 525s + 24y^3s + 333s^5x - 75x^2 + 117s^3x^2 + 18s^3y + 420s^2 - 161s^3 + 21s^5x^3 - 741xs^4 \\ + 12xy^2 - 13sx^3 - 518s^4 + 259s^5 + 16s^4y^3 - 270s^5y - 354s^4x^2 + 177s^2x^2 - 288s^2y \\ - 8s^5y^3 - 8s^3y^3 - 168s^4y^2 - 6x^2y + 540s^4y - 19x^3 + 8s^2x^3 - 204y^2s + 48s^2y^2 \\ + 36s^3y^2 + 144s^5x^2 + 570ys - 213sx^2 - 645sx + 15s^3x + 543s^2x + 36s^3x^3 - 57s^4x^3 + 84y^2s^5) \end{aligned}$$

As the Alexander polynomial of sextics of torus type with $8A_2$ is given $(t^2 - t + 1)^2$ ([22]), this explains $\Delta_{D_0}(t; L) = (t^2 - t + 1)^2$. Four inner A_2 's are not visible in the Figure 2. It is quite interesting to study how the family degenerates into $D_0 + 2L$. Observe that two cusps of D_0 are not real points and $L = \{x = 0\}$ is the bitangent line of D_0 . Note that (a-1) and (a-2) can not be a line-degenerated torus curve in the sense of the next subsection.

FIGURE 2. $C_s^{(3)}$, $s = -1/3$, sextics with $8A_2$

5.4. Line degeneration of curves of torus type. We consider a pair of coprime positive integers $p, q > 1$ and consider the curves of (p, q) -torus curve:

$$C_{p,q} : \quad f_p(X, Y, Z)^q + g_q(X, Y, Z)^p = 0$$

where f_p, g_q are polynomials of degree p, q respectively. Consider the special case that

$$f_p(X, Y, Z) = f_{p-a}(X, Y, Z) \times Z^a, \quad g_q(X, Y, Z) = g_{q-b}(X, Y, Z) \times Z^b$$

where f_{p-a}, g_{q-b} are homogeneous polynomials of degree $p-a, q-b$ respectively and $0 < a < p$ and $0 < b < q$. Assume for example that $aq \geq bp$ and factoring Z^{bp} from f , we have a curve

$$(3) \quad D : \quad g(X, Y, Z) = f_{p-a}(X, Y, Z)^q Z^{qa-pb} + g_{q-b}(X, Y, Z)^p = 0$$

We call a curve D a *line-degenerated torus curve of type (p, q)* and we call the line $L = \{Z = 0\}$ the *limit line of the degeneration*. Note that the degree of D is $pq - bp$. The simplest case is $a = b = 1$ and

$$D : \quad g(X, Y, Z) = f_{p-1}(X, Y, Z)^q Z^{q-p} + g_{q-1}(X, Y, Z)^p = 0, \quad p < q$$

Theorem 18. *For a degenerated torus curve D of type (p, q) defined (3), there is a family of line degeneration $f(X, Y, Z, t)$ of order bp such that $f_0(X, Y, Z) = g(X, Y, Z) Z^{bp}$ and each curve $C_t : f(X, Y, Z, t) = 0$ passes through a fixed point $Q \in L$ and the multiplicity of C_t at Q is pb for each $t \neq 0$.*

Proof. We may assume that $Q = (0, 1, 0)$ is not on D . Let $h(X, Y)$ be a homogeneous polynomial of degree $q-b$ with $h(0, Y) \neq 0$. We put

$$f_p(X, Y, Z) = f_{p-a}(X, Y, Z) Z^a, \quad g_q(X, Y, Z, t) = g_{q-b}(X, Y, Z) Z^b + t X^b h(X, Y)$$

and put

$$C_t : f(X, Y, Z, t) := f_p(X, Y, Z)^q + g_q(X, Y, Z, t)^p = 0$$

We can easily see that $C_t : f(X, Y, Z, t) = 0$ passes through Q and the multiplicity of (C_t, Q) is pb , as the local equation at Q is given by

$$f(x', 1, z', t) = f_{p-a}(x', 1, z')^q z'^{qa} + \left(g_{q-b}(x', 1, z') z'^b + t x'^b h(x', 1) \right)^p = 0$$

where $x' = X/Y, z' = Z/Y$. The affine equation of C_t in \mathbb{C}_L^2 is given by

$$f(x, y, 1, t) = f_{p-a}(x, y, 1)^q + \left(g_{q-b}(x, y, 1) + t x^b h(x, y) \right)^p = 0.$$

where $x = X/Z, y = Y/Z$. We see that $\deg_y f(x, y, 1, t) = pq - pb$. \square

Put $s = \gcd(p, q)$. As the generic Alexander polynomial of (p, q) -torus curve of degree pq is divisible by $\Delta_{p,q}(t) := (t^{pq/s} - 1)^s (t - 1) / ((t^p - 1)(t^q - 1))$, we get

Corollary 19. *Let D be as above and let $L = \{Z = 0\}$ be the limit line of the degeneration. Then $\Delta_D(t; L)$ is divisible by $\Delta_{p,q}(t)$.*

5.4.1. *Singularities of line-degenerated torus curves.* We consider the curve defined by (3)

$$D : \quad g(X, Y, Z) = f_{p-a}(X, Y, Z)^q Z^{qa-pb} + g_{q-b}(X, Y, Z)^p = 0$$

Suppose that $P \in D$ is a singular point. P is called *an inner singularity* (respectively *outer*) if $f_{p-a}(P) = g_{q-b}(P) = 0$ (resp. $g_{q-b}(P) \neq 0$). P is called *wild* if P is also on the limit line $Z = 0$. The following describes the type of inner non-wild singularities.

Lemma 20. ([5]) *Let C be a curve of torus type*

$$C : \quad f(x, y)^p + g(x, y)^q = 0, \quad p < q$$

and assume that $f(0, 0) = g(0, 0) = 0$ and the curves $f(x, y) = 0$ is smooth at O . Let ν be the local intersection number of $f(x, y) = g(x, y) = 0$ at O . Then the singularity (C, O) is topologically isomorphic to the Brieskorn singularity

$$B_{p, q\nu} : \quad y^p + x^{q\nu} = 0$$

The singularity is more complicated when $f(x, y) = 0$ is singular at O . The description of wild singularities is also more complicated in general.

5.5. Examples of line degeneration of torus curves.

5.5.1. *Cubic.* A cuspidal cubic Q can be understood as a line-degenerated (2,3)-torus curve of order 3 by taking

$$\begin{aligned} f_2(x, y) &= yx, \quad g_3(x, y) = g_1(x, y) x^2 \\ Q : \quad y^3 + g_1(x, y)^2 x &= 0. \end{aligned}$$

The limit line of the degeneration is $L = \{x = 0\}$. This explains that $\Delta_Q(t; L) = t^2 - t + 1$.

5.5.2. *Quartic.* We give further quartics which can be a line-degenerated (2,3)-torus curve. We consider the quartics of the form:

$$(4) \quad C_0 : \quad (f_2(x, y) x)^2 + (y x)^3 = 0$$

$$(5) \quad D : \quad g(x, y) = f_2(x, y)^2 + y^3 x = 0$$

where $f_2(x, y)$ is a polynomial of degree 2 and the limit line of degeneration is chosen to be $\{x = 0\}$. In general, D has two inner A_2 singularities at $y = f_2(x, y) = 0$. If $y = 0$ is tangent to the conic $C_2 := \{f_2(x, y) = 0\}$, the singularity is an A_5 . If moreover C_2 degenerates into two lines, the singularity is an E_6 -singularity. The limit line L is a bi-tangent line at $\{x = 0\} \cap C_2$. If L is tangent to the conic C_2 , D obtains a flex of order 2 and L is the flex tangent line. Further more we can put one outer singularity, either A_1 or A_2 . There are two more configurations which can be a line degeneration of torus curves: $A_2 + A_3$ and A_6 . For these singularities, we have to consider wild inner singularities. We have already studied most of these quartic and their Alexander invariants in §4.5. Theorem 14 and Theorem 18 explains our previous computations.

5.5.3. *A quartic with $2A_2$.* For a generic quartic D with $\Sigma(D) = 2A_2$, its dual curve \check{D} is a sextic with $8A_2 + A_1$, in particular, D has one bi-tangent line.

$$D : g(x, y) = (y^2 - 1 + x^2)^2 + y^3x = 0$$

The bi-tangent line can degenerate into a flex tangent line of order 2 so that \check{D} has $6A_2 + E_6$.

$$D : g(x, y) = (y^2 - 2y - 2x^2 + x + 1)^2 + y^3x = 0$$

See Figure 4, §5.9 for graphs of these quartics.

5.5.4. *A quartic with $2A_2 + A_1$.* A generic quartic D with $\Sigma(D) = 2A_2 + A_1$ has two flexes and one bi-tangent line (i.e., the configuration space is self dual). Degenerated quartic D' has one flexes of order 2 and no bi-tangent line as we have seen before.

$$D : \left(y^2 + \left(\frac{9}{2}x - 1 \right)y + \frac{3}{2}x^2 + x - 1 \right)^2 + y^3x = 0$$

$$D' : \left(y^2 + \left(\frac{17}{6}x - \frac{7}{3} \right)y + \frac{23}{6}x^2 - \frac{13}{3}x + 1 \right)^2 + y^3x = 0$$

See Figure 5, §5.9 for graphs of these quartics.

5.5.5. *A quartic with A_5 .* A generic quartic D with $\Sigma(D) = \{A_5\}$ has 6 flexes and one bi-tangent line, thus the dual curve is a sextic with $6A_2 + A_5 + A_1$. Degenerated quartic D' has 4 flexes and one flex of order 2 and thus the dual curve is a sextic with $4A_2 + E_6 + A_5$. Tokunaga has studied a certain dihedral covers branched along these quartics [33]

$$\begin{aligned} D &: (y^2 - yx - x^2 + 2x - 1)^2 + y^3x = 0 \\ D' &: (y^2 - 2y + x^2 + 2x + 1)^2 + y^3x = 0 \end{aligned}$$

See Figure 6, §5.9 for graphs of these quartics.

5.5.6. *A quartic with E_6 .* A generic quartic D with $\Sigma(D) = \{E_6\}$ has two flexes and one bi-tangent line. Degenerated one D' has one flexes of order 2 and no bi-tangent line as we have seen before.

$$(6) \quad D : (y^2 - 1 - x^2 + 2x)^2 + y^3x = 0 \quad D' : (x - y + 1)^4 + y^3x = 0$$

Note that the last quartic D' with E_6 can be also considered as a line degeneration of (3,4)-torus curves as

$$((x - y + 1)^4 + y^3x)x^8 = ((x - y + 1)x^2)^4 + (yx^3)^3$$

This explains that $\Delta_{D'}(t; L) = (t^2 - t + 1)(t^4 - t^2 + 1)$ with $L = \{x = 0\}$. See Figure 7, §5.9 for graphs of these quartics.

5.5.7. *quartics with $A_2 + A_3$ and A_6 .* We start the general form

$$C : f_2(x, y)^2 + y^3 x = 0$$

We assume that $O = (0, 0)$ is a wild inner singularity. Thus $f_2(0, 0) = 0$.

The configuration $A_2 + A_3$ is obtained when $f_2(0, 0) = 0$ and $f_2(x, y) = 0$ is not tangent to $y = 0$ as $(C, O) \cong A_3$. The limit line is tangent at one smooth point and also passing at O . For example,

$$C : (-y^2 + y - x^2 + x)^2 + y^3 x = 0$$

If we make $f_2(x, y) = 0$ is tangent to $y = 0$ at O , $(C, O) \cong A_6$. An example is given as follows.

$$C : (y^2 + y + x^2)^2 + y^3 x = 0$$

See Figure 8, §5.9 for graphs of these quartics.

5.5.8. *Quintics as line-degenerations.* We consider (2,5)-torus curve $C : f(x, y) = f_5(x, y)^2 + g_2(x, y)^5 = 0$ of degree 10 which is degenerated as

$$f_5(x, y) = f_2(x, y)x^3, g_2(x, y) = yx$$

Then we get a quintic

$$(7) \quad D : f_2(x, y)^2 x + y^5 = 0$$

In general, D has 2 A_4 singularities at $f_2(x, y) = y = 0$ and it has a flex of order 3 at O with the tangent line $L = \{x = 0\}$. As a special case where the conic $f_2(x, y) = 0$ is tangent to $y = 0$, we get one A_9 singularity:

$$D : (y^2 + xy + x^2 - 2x + 1)^2 x + y^5 = 0$$

If $f_2(x, y) = 0$ is two lines intersecting on $y = 0$, the singularity is locally topologically isomorphic to $C_{5,5}$ in the notation [30]:

$$C_{5,5} : y^5 + x^2 y^2 + x^5 = 0$$

If $f_2 = 0$ is a line with multiplicity 2, the singularity becomes $B_{4,5}$ singularity which is locally defined as $y^5 + x^4 = 0$.

There are two other possibilities. A quintic as a line degeneration of torus curves of type (3,5): take $g_3 = yx^2$, $f_5 = f_1(x, y)x^4$. Then we get a quintic

$$Q : y^5 + x^2 f_1(x, y)^3 = 0$$

The quintic Q has one E_8 singularity and a A_4 singularity on the limit line $x = 0$. The limit line is also the tangent line of the singularity A_4 .

Another possibility is as a line degeneration of torus curves of type (4,5): take $g_4 = yx^3$, $f_5 = g_1(x, y)x^4$. Then we get a quintic

$$Q' : y^5 + x g_1(x, y)^4 = 0$$

Q' has one $B_{5,4}$ -singularity and the limit line is $x = 0$. This quintic can be considered as a degeneration of (7) when $f_2(x, y) \rightarrow g_1(x, y)^2$.

5.6. Sextics as line degenerations. Sextics as line degenerations can be either from (2,5)-torus curves, or from (3,5)-torus curves or from (3,4)-torus curves. The sextics from (2,5)-torus curves take the form:

$$(8) \quad C : g(x, y) = f_3(x, y)^2 + y^5 x = 0.$$

Generically C has 3 A_4 singularities and the degeneration line $L = \{x = 0\}$ is a tri-tangent line. By the degeneration of the intersection $y = f_3(x, y) = 0$, we may have also either $A_4 + A_9$ or A_{14} . If the cubic $f_3(x, y) = 0$ has a node or cusp, the singularity becomes more complicated.

Sextics from (3,5) torus curves take the form:

$$(9) \quad C' : f_2(x, y)^3 + y^5 x = 0.$$

Generically C' has 2 $B_{3,5}$ singularities and degeneration line is a bi-tangent line at two flex points. If $y = 0$ is tangent to the conic $f_2(x, y) = 0$, the singularity is $B_{3,10}$.

The curves $C : f_2(x, y)^3 + y^4 x^2 = 0$ can be considered as a line degeneration of (3,4)-torus curves but at the same time, it is a torus curve of type (2,3). Generically C has $2E_6 + 2A_2$. Thus by Corollary 19, the Alexander polynomial $\Delta_C(t; L)$ is divisible by $(t^2 - t + 1)(t^4 - t^2 + 1)$. Note that $\Delta_C(t) = t^2 - t + 1$ by [27].

5.7. Flex degenerations. Let us consider C_τ is a family of irreducible curves in the configuration space $\mathcal{M}(\Sigma; d)$ with two marked flex points $P_\tau, Q_\tau \in C_\tau$ of order 1 for $\tau \neq 0$ and assume that

- (a) $P_\tau, Q_\tau \rightarrow P_0$ when $\tau \rightarrow 0$ and P_0 is a flex point of order 2 of $C_0 \in \mathcal{M}(\Sigma; d)$.
- (b) The intersection $T_{P_\tau} C_\tau \cap T_{Q_\tau} C_\tau \cap C_\tau$ is empty for $\tau \neq 0$.

Theorem 21. *Consider the degeneration: $C_\tau + L_{1,\tau} + L_{2,\tau} \rightarrow C_0 + 2L$. Then we have the divisibility of Alexander polynomials:*

$$\Delta_{C_\tau \cup L_{1,\tau} \cup L_{2,\tau}}(t) \mid \Delta_{C_0 \cup L^2}(t) \times (t - 1).$$

Here $L_{1,\tau} = T_{P_\tau} C_\tau$, $L_{2,\tau} = T_{Q_\tau} C_\tau$ and $L = T_{P_0} C_0$.

Proof. First we may assume that $C_\tau = \{f(x, y, \tau) = 0\}$, $P_0 = (0, 0)$ and the tangent line of C_0 at P_0 is defined by $y = 0$. Take a generic line L_∞ and we work in $\mathbb{C}^2 = \mathbb{C}_{L_\infty}^2$. Taking a presenting polydisk $\Delta_{\alpha, \beta}$ for $C_0 \cup L$ with pencil line L_η , $\eta \in \mathbb{C}$, let η_1, \dots, η_m be parameters corresponding to the singular pencil lines with $\eta_1 = 0$. Fix a small $\varepsilon > 0$ to see the monodromy relations along $|\eta - \eta_i| = \varepsilon$. We take generators in a fixed generic line L_{η_0} , $|\eta_0| = \varepsilon$ (see Appendix). Then we get a presentation:

$$(10) \quad \pi_1(\mathbb{C}^2 - C_0 \cup L) = \langle g_1, \dots, g_d, h \mid R_1, \dots, R_k \rangle$$

We consider the Alexander matrix M_0 with respect to the weight function θ_2 which has weight 2 for L . Take a positive number δ so that

$$C_\tau \pitchfork L_\eta, \quad \forall \tau, |\tau| \leq \delta, \forall \eta, |\eta - \eta_i| = \varepsilon, i = 1, \dots, m$$

We may also assume that $\Delta_{\alpha, \beta}$ is admissible for C_τ , $|\tau| \leq \delta$. Next, we consider $C_\tau \cup L_{1,\tau} \cup L_{2,\tau}$ for sufficiently small τ in the above sense. In the generic fiber L_η , the intersection $L_\eta \cap L$

are bifurcated into two points $L_\eta \cap L_{1,\tau}$ and $L_\eta \cap L_{2,\tau}$, but they are observed only with a microscope and they move exactly as a twin satellite along $|\eta - \eta_i| = \varepsilon$. For the presentation of $\pi_1(\mathbb{C}^2 - C_\tau \cup L_{2,\tau} \cup L_{2,\tau})$, we need two generators h_1, h_2 presented by lassos for the lines $L_{2,\tau}$ and $L_{2,\tau}$ instead of one h . However we can understand as $h = h_1 h_2$. For the further detail about the choice of generators, see Appendix. This implies that there are canonical homomorphisms ψ, Ψ which make the next diagram commutative.

$$\begin{array}{ccccc} F(d+1) & \longrightarrow & \pi_1(\mathbb{C}^2 - C_\tau \cup L) & \xrightarrow{\theta_2} & \mathbb{Z} \\ \downarrow \Psi & & \downarrow \psi & & \downarrow \text{id} \\ F(d+2) & \longrightarrow & \pi_1(\mathbb{C}^2 - C_\tau \cup L_{1,\tau} \cup L_{2,\tau}) & \xrightarrow{\theta_{sum}} & \mathbb{Z} \end{array}$$

Here Ψ is defined on generators as $g_i \mapsto g_i$, $h \mapsto h_1 h_2$ and ψ is canonically induced by Ψ . The monodromy relations R_1, \dots, R_k remains the same along $|\eta - \eta_j| = \varepsilon$. This means that the relation R'_j remains true where R'_j is obtained simply substituting the letter h by $h_1 h_2$. To get a complete relations, we have to add some more relations, say S_1, \dots, S_ℓ along the singular pencil lines. Among them, we can assume that

$$S_1 = g_d h_2 g_d^{-1} h_2^{-1}$$

which is the relation at the transverse intersection $L_{2,\tau} \cap C_\tau$ near P_0 . (In the appendix, we will explain this situation more.) Thus the presentation is given as

$$(11) \quad \pi_1(\mathbb{C}^2 - C_\tau \cup L_{1,\tau} \cup L_{2,\tau}) = \langle g_1, \dots, g_d, h_1, h_2 \mid R'_1, \dots, R'_k, S_1, \dots, S_\ell \rangle$$

Let G_∞ be the subgroup of $G_\tau := \pi_1(\mathbb{C}^2 - C_\tau \cup L_{1,\tau} \cup L_{2,\tau})$ generated by g_1, \dots, g_d and the product $h_1 h_2$. Then ψ is a surjection on $G_\infty \subset G_\tau$. For $C_\tau \cup L_{1,\tau} \cup L_{2,\tau}$, we consider the summation homomorphism θ_{sum} . Let γ_2, γ_τ be the ring homomorphisms corresponding to θ_2, θ_{sum} :

$$\gamma_2 : \mathbb{C}(F(d+1)) \rightarrow \mathbb{C}[t, t^{-1}], \quad \gamma_\tau : \mathbb{C}(F(d+2)) \rightarrow \mathbb{C}[t, t^{-1}]$$

Note that the following diagrams are commutative.

$$\begin{array}{ccccc} \gamma_2 : \mathbb{C}(F(d+1)) & \longrightarrow & \mathbb{C}(\pi_1(\mathbb{C}^2 - C_\tau \cup L)) & \xrightarrow{\theta_{2*}} & \mathbb{C}(\mathbb{Z}) = \Lambda \\ \downarrow \Psi_* & & \downarrow \psi_* & & \downarrow \text{id} \\ \gamma_\tau : \mathbb{C}(F(d+2)) & \longrightarrow & \mathbb{C}(\pi_1(\mathbb{C}^2 - C_\tau \cup L_{1,\tau} \cup L_{2,\tau})) & \xrightarrow{\theta_{sum*}} & \mathbb{C}(\mathbb{Z}) = \Lambda \end{array}$$

Now we consider the Alexander matrix M_τ of $C_\tau \cup L_{2,\tau} \cup L_{2,\tau}$. We consider the row corresponding to the relation R'_i . It is a word of g_1, \dots, g_d and $h_1 h_2$. By the definition of θ_2 , we can see easily that

$$\gamma_2\left(\frac{\partial R'_i}{\partial g_j}\right) = \gamma_\tau\left(\frac{\partial R'_i}{\partial g_j}\right)$$

The $(d+1)$ -th column split into two columns, which correspond to the Fox differentials $\frac{\partial}{\partial h_i}$, $i = 1, 2$. As $h = h_1 h_2$, we obtain

$$(12) \quad (\gamma_\tau\left(\frac{\partial R'_i}{\partial h_1}\right), \gamma_\tau\left(\frac{\partial R'_i}{\partial h_2}\right)) = (\gamma_2\left(\frac{\partial R_i}{\partial h}\right), t \times \gamma_2\left(\frac{\partial R_i}{\partial h}\right))$$

Note also that

$$(\gamma_\tau\left(\frac{\partial S_1}{\partial h_1}\right), \gamma_\tau\left(\frac{\partial S_1}{\partial h_2}\right)) = (0, t-1)$$

Let M'_τ be the matrix obtained by adding $(-t) \times (d+1)$ -th column to $(d+2)$ -th column so that the last column is zero up to k -th row. M'_τ is written as

$$\begin{pmatrix} M_0 & \vec{0} \\ \mathbf{w} & (t-1) \\ N_1 & \vec{v} \end{pmatrix}$$

where M_0 is the Alexander matrix of $C_0 \cup L$ and \mathbf{w} comes from the differential of S_1 , and the other terms N_1, \vec{v} are coming from $S_j, j \geq 2$. Thus for any $(d-1) \times (d-1)$ -minor A of M_0 , we associate $d \times d$ -minor A' of M'_τ , by adding $(k+1)$ -row and the last column. Then the corresponding determinant is equal to $\det(A) \times (t-1)$. Thus the assertion follows from the Fox calculus definition of the Alexander polynomial. \square

5.8. Appendix. In this appendix, we will explain the existence of the relation S_1 in the proof of Theorem 21. First we may assume that $C_\tau = \{f(x, y, \tau) = 0\}, P_0 = (0, 0)$ and the tangent line of C_0 at P_0 is defined by $y = 0$. Changing the scale and using Implicit function theorem, we may assume that C_0 is defined by $y = \phi_0(x)$ where $\phi_0(x) = x^4 + \text{higher terms}$ in the polydisk $\Delta_{1,1} = \{(x, y); |x|, |y| \leq 1\}$. This follows from the assumption that O is a flex of order 2 of C_0 . We consider the pencil lines $x = \eta$. Now we consider C_τ . Assume that C_τ is defined by $y = \phi_\tau(x)$ in $\Delta_{1,1}$. Write $\phi_\tau(x) = \sum_\nu c_\nu(\tau)x^\nu$. First observe that $|c_4(\tau) - 1| \ll 1$ by continuity. In the parametrization, the flex points are defined by $\{(\alpha, \beta) | \phi''_\tau(\alpha) = 0\}$. Thus by Rouché's principle, we see that there are two flex points which bifurcate from P_0 . They corresponds to the roots of $\phi''_\tau(x) = 0$ in $|x| \leq 1$, say $x = \alpha_1(\tau), \alpha_2(\tau)$. Thus $L_{1,\tau}, L_{2,\tau}$ corresponds to the tangent line at these flex points. By Bézout's theorem and the local stability of intersection numbers, there is one transverse intersection point of $L_{i,\tau} \cap C_\tau$ and we put them $Q_i = (\beta_i, \phi_\tau(\beta_i))$ for $i = 1, 2$. Note that $\beta_i \rightarrow 0$ when $\tau \rightarrow 0$. Thus the local singular pencils for $\tilde{C}_\tau = C_\tau \cup L_{1,\tau} \cup L_{2,\tau}$ is bifurcated in four points $x = \alpha_i, \beta_i, i = 1, 2$. We consider the local geometry of $p : (\Delta_{1,1}, \Delta_{1,1} \cap \tilde{C}_\tau) \rightarrow \Delta$. Let $\check{\phi}_\tau(x)$ be the polynomial of degree 4 which is the Taylor expansion of $\Phi_\tau(x)$ modulo x^5 . First we observe that this branched covering $p : (\Delta_{1,1}, \Delta_{1,1} \cap \tilde{C}_\tau) \rightarrow \Delta$ is topologically equivalent to the one where we replace C_τ by the curve $\check{C}_\tau = \{y = \check{\phi}_\tau(x)\}$. Next, the situation for \check{C}_τ and its two flex tangents inside $\Delta_{1,1}$ is equivalent to the following explicit one:

$$C'_\tau : y = \xi_\tau(x), \quad \xi_\tau(x) := x^4 - 6\tau^2 x^2$$

For this, we consider simply a homotopy $\Xi_t(x) = t\xi_\tau(x) + (1-t)\check{\phi}_\tau(x)$. Except a finite number of $t = t_1, \dots, t_q$, this family of curves defines equivalent covering over Δ . In this model $\xi_\tau(x)$, we have $\alpha_1, \alpha_2 = \pm\tau$ and $\beta_1, \beta_2 = \mp 3\tau$. We choose $\{x = 1\}$ as the fixed generic fiber i.e., $\eta_0 = 1$. On the fiber $x = 1$, P, Q_1, Q_2 are the intersections of the line $x = 1$ and C, L_1, L_2 respectively and we choose the generators as in Figure 7. The base point B is chosen on the circle $|y| = 1$. The other $d-1$ intersection points of $C_\tau \cap \{x = 1\}$ are outside of the unit disk and the generators g_1, \dots, g_{d-1} are omitted in the figure. The loops are oriented counterclockwise. Now we consider the loop in the base space $\ell \circ \omega \circ \ell^{-1}$ where ℓ is the line segment from $x = 1$ to $x = \beta_2 + \varepsilon'$, $\varepsilon' \ll (1 - \beta_2)$ and ω is the loop $|\eta - \beta_2| = \varepsilon'$. It is now

easy to see that the monodromy relation along this loop is nothing but $S_1 : g_d h_2 g_d^{-1} h_2^{-1}$ as is expected.

FIGURE 3. Choice of generators g_d, h_1, h_2

5.9. Graphs of various quartics. We put the graphs of various line-degenerated quartics of torus type.

FIGURE 4. Quartic with $2A_2$, bi-tangent limit line (left), flex tangent limit line (right)

FIGURE 5. Quartic with $2A_2 + A_1$, bi-tangent limit line (left), flex tangent limit line (right)

FIGURE 6. Quartic with A_5 , bi-tangent limit line (left), flex tangent limit line (right)

FIGURE 7. Quartic with E_6 , bi-tangent limit line (left), flex tangent limit line (right)

FIGURE 8. Quartic with $A_3 + A_2$ left, with A_6 right

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